UNIQUENESS AND STABILITY FOR THE SHOCK REFLECTION-DIFFRACTION PROBLEM FOR POTENTIAL FLOW

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Abstract. When a plane shock hits a two-dimensional wedge head on, it experiences a reflection-diffraction process, and then a self-similar reflected shock moves outward as the original shock moves forward in time. The experimental, computational, and asymptotic analysis has indicated that various patterns occur, including regular reflection and Mach reflection. The von Neumann conjectures on the transition from regular to Mach reflection involve the existence, uniqueness, and stability of regular shock reflection-diffraction configurations, generated by concave cornered wedges for compressible flow. In this paper, we discuss some recent developments in the study of the von Neumann conjectures. More specifically, we present our recent results of the uniqueness and stability of regular shock reflection-diffraction configurations governed by the potential flow equation in an appropriate class of solutions. We first show that the transonic shocks in the global solutions obtained in Chen-Feldman [19] are convex. Then we establish the uniqueness of global shock reflection-diffraction configurations with convex transonic shocks for any wedge angle larger than the detachment angle or the critical angle. Moreover, the stability of the solutions with respect to the wedge angle is also shown. Our approach also provides an alternative way of proving the existence of the admissible solutions established first in [19].

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1. **Introduction.** We survey some recent developments in the mathematical analysis of the shock reflection-diffraction problem for potential flow and the corresponding von Neumann conjectures on the existence, uniqueness, and stability of regular shock reflection-diffraction configurations for the transition from regular to Mach reflection. The shock reflection-diffraction problem is a lateral Riemann problem and has been not only longstanding open in fluid mechanics but also fundamental in the mathematical theory of multidimensional conservation laws.

When a planar shock hits a concave cornered wedge, the incident shock interacts with the wedge, leading to the occurrence of shock reflection-diffraction (*cf.* [12, 53]). Beginning from the work of E. Mach [45] in 1878, various patterns of shock reflection-diffraction configurations have been observed experimentally and later numerically, including regular reflection and Mach reflection. The existence of the regular reflection solutions for potential flow has been now fully understood mathematically (see [17, 19]), by reducing the shock reflection-diffraction problem to a *free boundary problem*, where the unknown reflected shock is regarded as a free boundary. Then a natural followup fundamental question is to study the uniqueness and stability of the solutions we have obtained.

For the uniqueness problem, it is necessary to restrict to a class of solutions. Recent results [24, 25, 33, 46] show the non-uniqueness of solutions with planar shocks in the class of entropy solutions with shocks of the Cauchy problem (initial value problem) for the multidimensional compressible Euler equations (isentropic and full). Our setup is different – the problem for solutions with shocks for potential flow is on the domain with boundaries, so these non-uniqueness results do not apply directly. These indicate that it is natural to study the uniqueness and stability problems in a more restricted class of solutions. In this paper, we show the uniqueness in the class of self-similar solutions of regular shock reflection-diffraction configurations with convex transonic shocks, which are called admissible solutions; see the detailed definition in §3. Technically, restricting to the class of admissible solutions allows us to reduce the uniqueness problem for shock reflection-diffraction to a corresponding uniqueness problem for solutions of a free boundary problem for a nonlinear elliptic equation, which is degenerate for the supersonic case (see Fig. 2.1 below).

A key property of admissible solutions which we employ in the uniqueness proof is that the admissible solutions converge to the unique normal reflection solution as the wedge angle tends to $\frac{\pi}{2}$. Then the outline of the uniqueness argument is the following: If there are two different admissible solutions, defined by the potential functions φ and φ^*, for some wedge angle $\theta_{\rm w}^* < \frac{\pi}{2}$, then it suffices to:

(i) construct continuous families of solutions parametrized by the wedge angle $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$, starting from φ and φ^*, respectively, in an appropriate norm;

(ii) prove *local uniqueness*: If two admissible solutions for the same wedge angle are close in the norm mentioned above, then they must be equal.

Combining this with the fact that both families converge to the unique normal reflection as $\theta_{\rm w} \to \frac{\pi}{2}-$, we conclude a contradiction.

Therefore, it remains to perform the two steps described above. Both steps can be achieved if we linearize the free boundary problem around an admissible solution, and then show that the linearization is sufficiently regular so that the solutions for close wedge angles can be constructed by the implicit function theorem. Indeed, this approach works for one regular shock reflection-diffraction case – the subsonic-away-from-sonic case (see §5 for more details).

However, it turns out that the linearization does not have such properties for the other case – the supersonic case, owing to the elliptic degeneracy near the sonic arc and relatively lower regularity of admissible solutions near the corner point between the shock and the sonic arc. For this case, instead, we develop a nonlinear approach: We prove directly the local uniqueness property and employ it to perturb any given admissible solution φ for the wedge angle $\theta_{\rm w}$, that is, to construct an admissible solution close to φ for all wedge angles which are sufficiently close to $\theta_{\rm w}$ by using the Leray-Schauder degree argument in a *small iteration set*. We note that, in [19], the solutions have also been constructed by the Leray-Schauder degree argument, but in a *large* iteration set, *i.e.* a subset in a space determined by some weighted and scaled $C^{k,\alpha}$ norms, with bounds by the constants sufficiently large so that the *apriori* estimates of the admissible solutions assure that a fixed point of the iteration map does not occur at the boundary of the iteration set. In the present case of *small* iteration set, the similar property is shown by using the local uniqueness.

Our proof of the local uniqueness is based on the convexity of the reflected-diffracted transonic shock, established in Chen-Feldman-Xiang [21]. We note that the convexity of the shocks is consistent with physical experiments and numerical simulations; see e.g. [4,12,26,28,34–39,42,47,50,52,54], and the references therein. Also see [10,11,41,43,44,48,50] for the convexity of transonic shocks in numerical Riemann solutions of the Euler equations for compressible fluids. Mathematically, the Rankine-Hugoniot conditions on the shock whose location is unknown, together with the nonlinear equation in the elliptic and hyperbolic regions, enforce a restriction to possible geometric shapes of the transonic shock. Moreover, one of our observations is that the convexity of transonic shocks is not a local property. In fact, the uniform convexity is a result of the interaction of the cornered wedge and the incident shock, since the reflected shock remains flat when the wedge is a flat wall for the normal shock reflection. In addition, for this problem, it seems to be difficult to apply the methods directly as in [5–7,29,49], owing to the difference and the more complicated structure of the boundary conditions.

In [21], we have developed an approach in which the global properties of solutions are incorporated in the proof of the convexity of transonic shocks. In particular, we have introduced a general set of conditions and employed the approach to prove the convexity of transonic shocks under these conditions. As a direct corollary, we have proved the uniform convexity of transonic shocks in the two longstanding fundamental shock problems – the shock reflection-diffraction problem by wedges and the reflection problem for supersonic flows past solid ramps.

Moreover, as a byproduct of our uniqueness proof, we have developed a new way of establishing the existence of global solutions of the shock reflection-diffraction problem up to the detachment angle or the critical angle, based on the fine convexity structure. Our approach is also helpful for other related mathematical problems including free boundary problems with degeneracy.

The previous works on unsteady flows with shocks in self-similar coordinates include the following: The problem of shock reflection-diffraction by a concave cornered wedges for potential flow has been systematically analyzed in Chen-Feldman [17,19] and Bae-Chen-Feldman [1], where the existence of regular shock reflection-diffraction configurations has been established up to the detachment wedge angle or the critical angle for potential flow. For the Mach reflection, S. Chen [23] proved the local stability of flat Mach configuration in self-similar coordinates. Also see [8, 9, 16, 27, 40] for the unsteady transonic small disturbance equation and the nonlinear wave system, [51] for the Chaplygin gas, and [56] for the pressure-gradient system. Meanwhile, other problems have been tackled. For the shock diffraction problem, Chen-Feldman-Hu-Xiang [20] showed that regular shock configurations cannot exist for potential flow. For supersonic flow past a solid ramp, Elling-Liu [30] obtained a first rigorous unsteady result under certain assumptions for potential flow. Then Bae-Chen-Feldman [2,3] succeeded to remove the assumptions in [30] and established the existence theorem for global shock reflection configurations so that the steady supersonic weak shock solution as the long-time behavior of an unsteady flow for all physical parameters, via new mathematical techniques developed first in Chen-Feldman [19]. See also [13–15, 31, 32] and the references therein for the steady transonic shocks over two-dimensional wedges.

The organization of this paper is the following: In §2, we introduce the free boundary problem for the shock reflection-diffraction problem. Then the existence and regularity results obtained in [19] are given in §3. In §4, we describe the result and present the main steps of the proof on the convexity of the regular reflected-diffracted transonic shock based on [21]. In §5, we discuss our recent result and outline the proof on the uniqueness and stability of regular shock reflection-diffraction configurations.

2. The Potential Flow Equation and the Shock Reflection-Diffraction Problem.

In this section we formulate the shock reflection-diffraction problem as a free boundary problem for the potential flow equation in the self-similar coordinates.

2.1. The potential flow equation.

The Euler equations for potential flow consist of the conservation law of mass and Bernoulli's law:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \tag{2.1}$$

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, \tag{2.2}$$

where ρ is the density, Φ is the velocity potential so that $\mathbf{v} = \nabla_{\mathbf{x}} \Phi$, B_0 is the Bernoulli constant determined by the incoming flow and/or boundary conditions, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, and $i(\rho) = \int_1^\rho \frac{p'(s)}{s} \mathrm{d}s$ for the pressure function $p = p(\rho)$. For polytropic gas, by scaling,

$$p(\rho) = \frac{\rho^\gamma}{\gamma}, \quad c^2(\rho) = \rho^{\gamma-1}, \quad i(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}, \quad \gamma > 1,$$

where $c(\rho)$ is the sound speed.

The system is invariant under the self-similar scaling:

$$(\mathbf{x}, t) \to (\alpha \mathbf{x}, \alpha t), \quad (\rho, u, v, \Phi) \to (\rho, u, v, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0.$$

Thus, we can seek self-similar solutions of the form:

$$(\rho, u, v)(\mathbf{x}, t) = (\rho, u, v)(\boldsymbol{\xi}), \quad \Phi(\mathbf{x}, t) = t\big(\varphi(\boldsymbol{\xi}) + \frac{1}{2}|\boldsymbol{\xi}|^2\big) \qquad \text{for } \boldsymbol{\xi} = (\xi_1, \xi_2) = \tfrac{\mathbf{x}}{t},$$

where φ is called a pseudo–velocity potential that satisfies $\nabla_{\boldsymbol{\xi}} \varphi = (u - \xi_1, v - \xi_2) = (U, V)$ which is called a pseudo-velocity. Then the pseudo–potential function φ satisfies the following equation for self–similar solutions:

$$\operatorname{div}(\rho D\varphi) + 2\rho = 0, \tag{2.3}$$

where the density function $\rho = \rho(|D\varphi|^2, \varphi)$ is determined by

$$\rho(|D\varphi|^2, \varphi) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2)\big)^{\frac{1}{\gamma-1}}, \tag{2.4}$$

and the divergence *div* and gradient D are with respect to the self–similar variables $\boldsymbol{\xi}$, and ρ_0 is a positive constant (to be given in Problem 2.1 below) so that $\rho_0^{\gamma-1} = (\gamma-1)B_0 + 1$. Therefore, the potential function φ is governed by the following second-order potential flow equation:

$$div\big(\rho(|D\varphi|^2, \varphi)D\varphi\big) + 2\rho(|D\varphi|^2, \varphi) = 0. \tag{2.5}$$

Equation (2.5) is a second-order equation of mixed hyperbolic-elliptic type: It is elliptic if and only if $|D\varphi| < c(|D\varphi|^2, \varphi)$, which is equivalent to

$$|D\varphi| < c_\star(\varphi, \gamma) := \sqrt{\frac{2}{\gamma+1}\big(\rho_0^{\gamma-1} - (\gamma-1)\varphi\big)}. \tag{2.6}$$

If ρ is a constant, then (2.3)–(2.4) implies that the corresponding pseudo-velocity potential φ is of the form:

$$\varphi(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u, v)\cdot\boldsymbol{\xi} + k$$

for constants u, v, and k. Such a solution is called a uniform or constant state.

2.2. **Weak solutions and the Rankine-Hugoniot conditions.** Since shocks are involved in the problem under consideration, we define the notion of weak solutions of equation (2.5), which admits the shocks.

Definition 2.1. A function $\varphi \in W^{1,1}_{loc}(\Omega)$ is called a weak solution of (2.5) if

(i) $\rho_0^{\gamma-1} - \varphi - \frac{1}{2}|D\varphi|^2 \ge 0 \quad a.e.$ in Ω,

(ii) $(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in (L^1_{loc}(\Omega))^2$,

(iii) For every $\zeta \in C^\infty_c(\Omega)$,

$$\int_\Omega \big(\rho(|D\varphi|^2, \varphi)D\varphi\cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta\big)\mathrm{d}\boldsymbol{\xi} = 0.$$

For a piecewise smooth solution φ divided by a shock, it is easy to verify that φ satisfies the conditions in Definition 2.1 if and only if φ is a classic solution of (2.5) in each smooth subregion and satisfies the following Rankine-Hugoniot conditions across the shock:

$$[\rho(|D\varphi|^2, \varphi)D\varphi\cdot\boldsymbol{\nu}]_S = 0, \tag{2.7}$$

$$[\varphi]_S = 0, \tag{2.8}$$

where $\boldsymbol{\nu}$ is a unit normal to S. Condition (2.7) is due to the conservation of mass, while condition (2.8) is due to the irrotationality.

There are fairly many weak solutions to the given conservation laws. The physically relevant solutions must satisfy the entropy condition. For potential flow, the discontinuity of $D\varphi$ satisfying the Rankine-Hugoniot conditions (2.7)–(2.8) is called a shock if it satisfies the following *entropy condition*: *The density ρ increases across a shock in the pseudo–flow direction.* From (2.7), the entropy condition indicates that the normal derivative function $\varphi_{\boldsymbol{\nu}} = D\varphi\cdot\boldsymbol{\nu}$ on a shock always decreases across the shock in the pseudo–flow direction.

2.3. **Shock reflection-diffraction problem.** The incident shock separates two constant states: state (0) with density ρ_0 and velocity $\mathbf{v}_0 = (0,0)$ ahead of the shock, and state (1) with density ρ_1 and velocity $\mathbf{v}_1 = (u_1, 0)$ behind the shock, where the entropy condition holds: $\rho_1 > \rho_0$ on the shock. The incident shock moves from the left to the right and hits the vertex of wedge:

$$W := \{\mathbf{x} : |x_2| < x_1 \tan\theta_{\rm w}, x_1 > 0\}$$

at the initial time. The slip boundary condition $\mathbf{v}\cdot\boldsymbol{\nu} = 0$ is prescribed on the solid wedge boundary.

Then the shock reflection-diffraction problem can be formulated as follows:

Problem 2.1 (Initial-boundary value problem)**.** *Seek a solution of system* (2.1)–(2.2) *for* $B_0 = \frac{\rho_0^{\gamma-1}-1}{\gamma-1}$ *with the initial condition at* $t=0$:

$$(\rho,\Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \text{for } |x_2| > x_1\tan\theta_{\rm w},\ x_1 > 0, \\ (\rho_1, u_1x_1) & \text{for } x_1 < 0, \end{cases} \tag{2.9}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla_{\mathbf{x}}\Phi\cdot\boldsymbol{\nu}|_{\partial W\times\mathbb{R}_+} = 0, \tag{2.10}$$

where $\boldsymbol{\nu}$ *is the exterior unit normal to* ∂W.

The initial-boundary value problem, Problem 2.1, is a lateral Riemann problem with boundary $\partial W\times\mathbb{R}_+$ in the $(\mathbf{x}, t)$–coordinates. Since state (1) does not satisfy the slip boundary condition, the solution must differ from state (1) behind the shock so that the shock reflection-diffraction configurations occur. These configurations are self-similar, so the problem can be reformulated in the self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Depending on the data, there may be various patterns of shock reflection-diffraction configurations, including regular reflection and Mach reflection.

By the symmetry of the problem with respect to the ξ_1–axis, we consider only the upper half-plane $\{\xi_2 > 0\}$ and prescribe the condition $\varphi_{\boldsymbol{\nu}} = 0$ on the symmetry line $\{\xi_2 > 0\}$. Note that state (1) satisfies this condition.

We study self-similar solutions of Problem 2.1. Thus we give a formulation for the solution of Problem 2.1 in self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Let

$$\Lambda = \mathbb{R}^2_+ \setminus \{\boldsymbol{\xi} : \xi_1 > 0,\ 0 < \xi_2 < \xi_1\tan\theta_{\rm w}\},$$

where $\mathbb{R}^2_+ = \mathbb{R}^2 \cap \{\xi_1 > 0\}$. Then, following Definition 2.1, we have

Definition 2.2. $\varphi \in C^{0,1}(\overline{\Lambda})$ is a weak solution of the shock reflection-diffraction problem if φ satisfies equation (2.5) in Λ in the weak sense, the boundary condition:

$$\partial_{\boldsymbol{\nu}}\varphi = 0 \qquad \text{on } \partial\Lambda, \tag{2.11}$$

and the asymptotic condition:

$$\lim_{R\to\infty} \|\varphi - \overline{\varphi}\|_{0,\Lambda\setminus B_R(0)} = 0, \tag{2.12}$$

where

$$\bar{\varphi} = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0,\ \xi_2 > \xi_1\tan\theta_{\rm w}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0,\ \xi_2 > 0, \end{cases}$$

and $\xi_1^0 > 0$ is the location of the incident shock.

2.4. **Solutions of regular reflection structure.** We will show that, for certain values of parameters, there exist self-similar solutions of the regular reflection structure for the shock reflection-diffraction problem and, moreover, these solutions are unique in the class of self-similar solutions of such a structure.

Figs. 2.1–2.2 show two different regular shock reflection-diffraction configurations in the self-similar coordinates. The regular reflection solutions are piecewise smooth; more precisely, they are smooth away from the incident and reflected-diffracted shocks, as well as the sonic circle (which is a weak discontinuity) for the supersonic case as shown in Fig. 2.1.

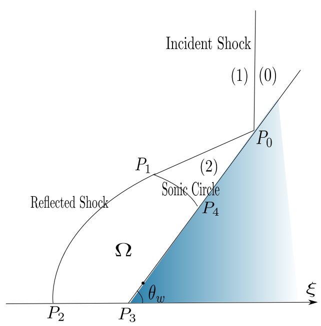

FIGURE 2.1. Supersonic regular shock reflection-diffraction configuration

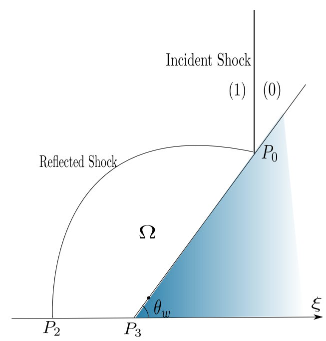

FIGURE 2.2. Subsonic regular shock reflection-diffraction configuration

A necessary condition for the existence of piecewise-smooth regular shock reflection-diffraction configurations is the existence of the constant state (2) with the pseudo-potential φ_2 that satisfies both the slip boundary condition on the wedge and the Rankine-Hugoniot conditions with state (1) across the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$, which passes through point P_0 where the incident shock meets the wedge boundary. Therefore, it requires the following three conditions at P_0:

$$\begin{aligned} &D\varphi_2 \cdot \boldsymbol{\nu}_{\mathrm{w}} = D\varphi_1 \cdot \boldsymbol{\nu}_{\mathrm{w}}, \\ &\varphi_2 = \varphi_1, \\ &\rho(|D\varphi_2|^2, \varphi_2) D\varphi_2 \cdot \boldsymbol{\nu}_{S_1} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu}_{S_1}, \end{aligned} \tag{2.13}$$

where $\boldsymbol{\nu}_{\mathrm{w}}$ is the outward normal to the wedge boundary, and $\boldsymbol{\nu}_{S_1} = \frac{D(\varphi_1-\varphi_2)}{|D(\varphi_1-\varphi_2)|}$.

It is well-known (see *e.g.* [19]) that, for given parameters (ρ_0, ρ_1) of states (0) and (1), there exists a detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}} \in (0, \frac{\pi}{2})$ such that the algebraic equations (2.13) have two solutions for each wedge angle $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, which become equal when $\theta_{\mathrm{w}} = \theta_{\mathrm{w}}^{\mathrm{d}}$. Then two two-shock configurations occur at P_0 when $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$. For each θ_{w}, state (2) with the smaller density is called a weak state (2). In this paper, state (2) always refers to the weak one, since the strong state (2) is ruled out by the stability/continuity criterion as introduced first by Chen-Feldman in [17]; see also [19]. Depending on the wedge angle, state (2) can be either supersonic or subsonic at P_0. Moreover, for θ_{w} near $\frac{\pi}{2}$ (resp. for θ_{w} near $\theta_{\mathrm{w}}^{\mathrm{d}}$), state (2) is supersonic (resp. subsonic) at P_0. The type of state (2) at P_0 determines the type of reflection, *i.e.* supersonic or subsonic, as shown in Figs. 2.1–2.2.

We consider solutions of the structure shown in Figs. 2.1–2.2. Outside of region Ω, the flow consists of the uniform states (0), (1), and (2) as indicated on the pictures, separated by the straight shocks. Flow is non-uniform and pseudo-subsonic in Ω. Here Ω is an open bounded connected domain, and $\partial\Omega = \overline{\Gamma_{\rm shock}} \cup \overline{\Gamma_{\rm sonic}} \cup \overline{\Gamma_{\rm wedge}} \cup \overline{\Gamma_{\rm sym}}$, where curve $\Gamma_{\rm shock}$ with endpoints P_1 and $P_2 \in \{\xi_2 = 0\}$ in the supersonic case (resp. P_0 and $P_2 \in \{\xi_2 = 0\}$ in the subsonic case) is a transonic shock which separates a constant state (1) outside Ω from a pseudo-subsonic (non-constant) state inside Ω, and $\overline{\Gamma_{\rm sonic}} \cup \overline{\Gamma_{\rm wedge}} \cup \overline{\Gamma_{\rm sym}}$ is the fixed boundary with arc $\Gamma_{\rm sonic}$ between points P_1 and P_4 of the pseudo-sonic circle of state (2) (we also use notation $\overline{\Gamma_{\rm sonic}} = \{P_0\}$ for the subsonic reflection case as shown in Fig. 2.2), the line segment $\Gamma_{\rm wedge}$ is the part of $\partial\Omega$ on the wedge boundary, *i.e.* $\Gamma_{\rm wedge} = P_3P_4$ in the supersonic case and $\Gamma_{\rm wedge} = P_0P_3$ in the subsonic case, and $\Gamma_{\rm sym} = P_2P_3$ is the part of $\partial\Omega$ on the symmetry line $\{\xi_2 = 0, \xi_1 < 0\}$.

3. **Existence and Regularity of Regular Shock Reflection-Diffraction Configurations.** We first notice that a key obstacle to the existence of regular shock reflection-diffraction configurations is an additional possibility that, at the critical wedge angle $\theta_{\rm w}^{\rm c} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, the reflected shock P_0P_2 may attach to the wedge vertex P_3, *i.e.* $P_2 = P_3$. We can rule out such a solution if $u_1 \le c_1$. In the opposite case $u_1 > c_1$, there would be a possibility that the reflected shock is attached to the wedge vertex, as the experiments show (*e.g.* [53, Fig. 238]). We note that the condition on (u_1, c_1) can be explicitly expressed through parameters (ρ_0, ρ_1) of states (0) and (1), besides $\gamma \ge 1$, by using (2.4) and the Rankine-Hugoniot conditions on the incident shock. Recall that $\rho_1 > \rho_0$. It can be shown that there exists $\rho^c > \rho_0$ such that

$$u_1 \le c_1 \quad \text{iff } \rho_1 \in [\rho_0, \rho^c], \qquad u_1 > c_1 \quad \text{iff } \rho_1 \in [\rho^c, \infty).$$

Now we state the existence and regularity results for the solutions of shock reflection-diffraction problem which have regular reflection structure as on Fig. 2.1–2.2, established in Chen-Feldman [19]. We prove these results in the class of *admissible solutions* of regular reflection problem, defined as following:

Definition 3.1. Let $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. A function $\varphi \in C^{0,1}(\overline{\Lambda})$ is an admissible solution of the regular reflection problem (2.5) and (2.11)–(2.12) if φ is a solution in the sense of Definition 2.2 and satisfies the following properties:

(i) The structure of solutions is as follows:

- If $|D\varphi_2(P_0)| > c_2$, then φ is of the *supersonic* regular shock reflection-diffraction configuration described in §2.4 and shown on Fig. 2.1 and satisfies:
  The reflected-diffracted shock $\Gamma_{\rm shock}$ is C^2 in its relative interior, and curve $P_0P_1P_2$ is C^1 up to its endpoints. Curves $\Gamma_{\rm shock}$, $\Gamma_{\rm sonic}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm symm}$ do not have common points except their endpoints.
  φ satisfies the following properties:

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \setminus P_0P_1P_2),$$
$$\varphi \in C^1(\overline{\Omega}) \cap C^3(\overline{\Omega} \setminus (\overline{\Gamma_{\rm sonic}} \cup \{P_2, P_3\})),$$

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\mathrm{w}}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ \varphi_2 & \text{in region } P_0P_1P_4, \end{cases} \tag{3.1}$$

(3.2)

- If $|D\varphi_2(P_0)| \leq c_2$, then φ is of the *subsonic* regular shock reflection-diffraction configuration described in §2.4 and shown on Fig. 2.2 and satisfies:
  The reflected-diffracted shock Γ_{shock} is C^2 in its relative interior, and is C^1 up to its endpoints. Curves Γ_{shock}, Γ_{wedge}, and Γ_{symm} do not have common points except their endpoints.
  φ satisfies the following properties:

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \setminus \Gamma_{\text{shock}}),$$
$$\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^3(\overline{\Omega} \setminus \{P_0, P_3\}),$$

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\mathrm{w}}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_2, \\ \varphi_2(P_0) & \text{at } P_0, \end{cases} \tag{3.3}$$

$$D\varphi(P_0) = D\varphi_2(P_0).$$

Moreover, in both supersonic and subsonic cases, denote $\Gamma^{\text{ext}}_{\text{shock}} = \Gamma_{\text{shock}} \cup \{P_0\} \cup \Gamma^-_{\text{shock}}$, where Γ^-_{shock} is the reflection of Γ_{shock} with respect to the ξ_1-axis. Then curve $\Gamma^{\text{ext}}_{\text{shock}}$ is C^1 in its relative interior.

(ii) Equation (2.5) is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}}$, *i.e.*

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}},$$

where, for the subsonic and sonic cases, we use notation $\overline{\Gamma_{\text{sonic}}} = \{P_0\}$.

(iii) $\partial_{\boldsymbol{\nu}}\varphi_1 > \partial_{\boldsymbol{\nu}}\varphi > 0$ on Γ_{shock}, where $\boldsymbol{\nu}$ is the normal to Γ_{shock} interior for Ω.

(iv) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω.

(v) Let $\mathbf{e}_{S_1}$ be the unit vector parallel to $S_1 := \{\varphi_1 = \varphi_2\}$, oriented so that $\mathbf{e}_{S_1} \cdot D\varphi_2(P_0) > 0$:

$$\mathbf{e}_{S_1} = -\frac{(v_2,\, u_1 - u_2)}{\sqrt{(u_1-u_2)^2 + v_2^2}}. \tag{3.4}$$

Let $\mathbf{e}_{\xi_2} = (0,1)$. Then

$$\partial_{\mathbf{e}_{S_1}}(\varphi_1 - \varphi) \leq 0, \quad \partial_{\xi_2}(\varphi_1 - \varphi) \leq 0 \qquad \text{on } \Gamma_{\text{shock}}. \tag{3.5}$$

Remark 3.1. It can be shown that Definition 3.1 is equivalent to the definition of admissible solutions in [19]; see Definitions 15.1.1–15.1.2 there. Thus, all the estimates and properties of admissible solutions shown in [19] hold for the admissible solutions defined above. In particular, the admissible solutions converge (in an appropriate sense) to the normal reflection solution as $\theta_{\mathrm{w}} \to \frac{\pi}{2}-$.

Remark 3.2. For the supersonic case, $\mathbf{e}_{S_1}$ defined by (3.4) has the expression:

$$\mathbf{e}_{S_1} = \frac{P_1 - P_0}{|P_1 - P_0|}.$$

Moreover, in the supersonic (resp. subsonic/sonic) case, $\mathbf{e}_{S_1}$ is tangential to Γ_{shock} in its upper endpoint P_1 (resp. P_0), because $(\varphi, D\varphi)|_\Omega = (\varphi_2, D\varphi_2)$ at that point, and its orientation at that endpoint of Γ_{shock} is towards the relative interior of Γ_{shock}.

Remark 3.3. Since the admissible solution φ is a weak solution in the sense of Definition 2.2 and is of regularity as in (i) of Definition 3.1, it satisfies (2.5) classically in Ω with the Rankine-Hugoniot conditions:

$$\varphi = \varphi_1, \quad \rho(|D\varphi|^2, \varphi) D\varphi \cdot \boldsymbol{\nu} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu} \qquad \text{on } \Gamma_{\text{shock}}, \tag{3.6}$$

and the boundary condition:

$$\partial_{\boldsymbol{\nu}} \varphi = 0 \qquad \text{on } \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}. \tag{3.7}$$

Remark 3.4. The admissible solution φ is not a constant state in Ω. Indeed, if φ is a constant state in Ω, then $\varphi = \varphi_2$ in Ω: This follows from both (3.1) for the supersonic case (since φ is $C^{1,1}$ across Γ_{sonic}) and property $(\varphi, D\varphi) = (\varphi_2, D\varphi_2)$ at P_0 for the subsonic case. However, φ_2 does not satisfy (3.7) on Γ_{sym} since $\mathbf{v}_2 = (u_2, v_2) = (u_2, u_2 \tan\theta_{\rm w})$ with $u_2 > 0$ and $\theta_{\rm w} \in (0, \frac{\pi}{2})$.

The following theorem shows that the admissible solution has additional regularity and monotonicity properties.

Theorem 3.1 (Properties of admissible solutions)**.** *There exits a constant $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, \frac{1}{2})$ such that any admissible solution in the sense of Definition 3.1 with wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ has the following properties*:

(i) *Additional regularity*:
- *If $|D\varphi_2(P_0)| > c_2$, i.e. when φ is of the* supersonic *regular shock reflection-diffraction configuration as in Fig. 2.1, it satisfies $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus (\overline{\Gamma_{\text{shock}}} \cup \{P_3\}))$, and φ is $C^{1,1}$ across Γ_{sonic}, including endpoints P_1 and P_4. The reflected-diffracted shock $P_0P_1P_2$ is $C^{2,\beta}$ up to its endpoints for any $\beta \in [0, \frac{1}{2})$, and C^∞ except P_1.*
- *If $|D\varphi_2(P_0)| \le c_2$, i.e. when φ is of the* subsonic *regular shock reflection-diffraction configuration as in Fig. 2.2, it satisfies*
$$\varphi \in C^{1,\beta}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega} \setminus \{P_0\}) \cap C^\infty(\overline{\Omega} \setminus \{P_0, P_3\})$$
for some $\beta = \beta(\rho_0, \rho_1, \gamma, \theta_w) \in (0, \alpha]$ where β is non-decreasing with respect to θ_w, and the reflected-diffracted shock Γ_{shock} is $C^{1,\beta}$ up to its endpoints and C^∞ except P_0.

(ii) *For each $\mathbf{e} \in Con(\mathbf{e}_{S_1}, \mathbf{e}_{\xi_2})$,*
$$\partial_{\mathbf{e}}(\varphi_1 - \varphi) < 0 \qquad \text{in } \Omega, \tag{3.8}$$
where vectors $\mathbf{e}_{S_1}$ $\mathbf{e}_{\xi_2}$ are defined in Definition 3.1(v), and
$$Con(\mathbf{e}_{S_1}, \mathbf{e}_{\xi_2}) := \{a\mathbf{e}_{S_1} + b\mathbf{e}_{\xi_2} \ : \ a, b > 0\}. \tag{3.9}$$

(iii) *For the supersonic reflection-diffraction configuration as in Fig. 2.1, the following regularity near Γ_{sonic} holds: $\varphi \in C^{2,\alpha}(\Omega \cup (\overline{\Gamma_{\text{sonic}}} \setminus \{P_1\})$ and, for any $\boldsymbol{\xi}^0 \in \overline{\Gamma_{\text{sonic}}} \setminus \{P_1\}$,*
$$\lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0\\ \boldsymbol{\xi}\in\Omega}} D_{rr}(\varphi - \varphi_2) = \frac{1}{\gamma+1}, \quad \lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0\\ \boldsymbol{\xi}\in\Omega}} D_{r\theta}(\varphi - \varphi_2) = \lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0\\ \boldsymbol{\xi}\in\Omega}} D_{\theta\theta}(\varphi - \varphi_2) = 0,$$
where (r, θ) are the polar coordinates with center at $O_2 = (u_2, v_2)$.

Remark 3.5. $Con(\mathbf{e}_{S_1}, \mathbf{e}_\eta) = \{a\mathbf{e}_{S_1} + b\mathbf{e}_\eta : a, b > 0\}$ is an open set; that is, it does not include the directions of $\mathbf{e}_{S_1}$ and $\mathbf{e}_{\xi_2}$.

Now we state the results on the existence of admissible solutions.

Theorem 3.2 (Global solutions up to the detachment angle for the case: $u_1 \leq c_1$)**.** *Let the initial data (ρ_0, ρ_1, γ) satisfy that $u_1 \leq c_1$. Then, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exists an admissible solution of the regular reflection problem in the sense of Definition 3.1. Note that these solutions satisfy the properties stated in Theorem 3.1.*

Theorem 3.3 (Global solutions up to the detachment angle for the case: $u_1 > c_1$)**.** *Let the initial data (ρ_0, ρ_1, γ) satisfy that $u_1 > c_1$. Then there is $\theta_{\rm w}^{\rm c} \in [\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ such that, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$, there exists an admissible solution of the regular reflection problem in the sense of Definition 3.1. Note that these solutions satisfy the properties stated in Theorem 3.1.*

If $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm d}$, then, for the wedge angle $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$, there exists an attached shock solution φ with all the properties listed in Definition 3.1 and Theorem 3.1(ii)–(iii) except that $P_2 = P_3$. In addition, for the regularity of solution φ, we have

- *For the supersonic case with $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$,*
$$\varphi \in C^\infty(\overline{\Omega}\backslash(\Gamma_{\rm sonic} \cup \{P_3\})) \cap C^{1,1}(\overline{\Omega}\backslash\{P_3\}) \cap C^{0,1}(\overline{\Omega}),$$
and the reflected shock $P_1P_2P_3$ is Lipschitz up to the endpoints, $C^{2,\beta}$ with any $\beta \in [0, \frac{1}{2})$ except point P_3, and C^∞ except points P_1 and P_3.
- *For the subsonic case with $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$,*
$$\varphi \in C^\infty(\overline{\Omega}\backslash\{P_1,\, P_3\}) \cap C^{1,\beta}(\overline{\Omega}\backslash\{P_3\}) \cap C^{0,1}(\overline{\Omega}),$$
for β as in Theorem 3.1, and the reflected shock $P_1P_2P_3$ is Lipschitz up to the endpoints, $C^{1,\beta}$ except point P_3, and C^∞ except points P_1 and P_3.

In the next two sections, §4–§5, we show how the convexity of the transonic shocks and the uniqueness of the admissible solutions can be achieved.

4. **Convexity of Transonic Shocks in the Shock Reflection-Diffraction Configurations.**

We first note that, for an admissible solution, $\Gamma_{\rm shock}$ is a graph in any direction $\mathbf{e} \in Con := Con(\mathbf{e}_{S_1}, \mathbf{e}_\eta)$, where $Con(\mathbf{e}_{S_1}, \mathbf{e}_\eta)$ is defined in (3.9). For the subsonic/sonic reflections case, we denote $P_1 := P_0$ so that $\Gamma_{\rm shock}$ has endpoints P_1 and P_2 in all cases. More precisely, the following was shown in [19], as a consequence of Theorem 3.1(ii):

Lemma 4.1. *Let φ be an admissible solution. Denote $\phi := \varphi - \varphi_1$. Let $\boldsymbol{\tau}_{P_1}$ be a unit tangent vector to $\Gamma_{\rm shock}$ at P_1, directed into the interior of $\Gamma_{\rm shock}$. Let $\mathbf{e} \in Con$, and let $\mathbf{e}^\perp$ be the orthogonal unit vector to $\mathbf{e}$ with $\mathbf{e}^\perp \cdot \boldsymbol{\tau}_{P_1} > 0$. Let (S, T) be the coordinates with respect to basis $\{\mathbf{e}, \mathbf{e}^\perp\}$ so that $T_{P_2} > T_{P_1}$. Then there exists $f_{\mathbf{e}} \in C^1(\mathbb{R})$ such that*

(a) $\Gamma_{\rm shock} = \{S = f_{\mathbf{e}}(T) : T_{P_1} < T < T_{P_2}\}$, $\Omega \subset \{S < f_{\mathbf{e}}(T) : T \in \mathbb{R}\}$, $P_1 = (f_{\mathbf{e}}(T_{P_1}), T_{P_1})$, $P_2 = (f_{\mathbf{e}}(T_{P_2}), T_{P_2})$, *and* $f_{\mathbf{e}} \in C^\infty(T_{P_1}, T_{P_2})$;

(b) *The directions of the tangent lines to $\Gamma_{\rm shock}$ lie between $\boldsymbol{\tau}_{P_1}$ and $\boldsymbol{\tau}_{P_2}$; that is, in the (S, T)–coordinates,*
$$-\infty < \frac{\boldsymbol{\tau}_{P_2} \cdot \mathbf{e}}{\boldsymbol{\tau}_{P_2} \cdot \mathbf{e}^\perp} = f'_{\mathbf{e}}(T_{P_2}) \leq f'_{\mathbf{e}}(T) \leq f'_{\mathbf{e}}(T_{P_1}) = \frac{\boldsymbol{\tau}_{P_1} \cdot \mathbf{e}}{\boldsymbol{\tau}_{P_1} \cdot \mathbf{e}^\perp} < \infty$$

for any $T \in (T_{P_1}, T_{P_2})$;

(c) $\boldsymbol{\nu}(P) \cdot \mathbf{e} < 0$ *for any* $P \in \Gamma_{\text{shock}}$;

(d) $\phi_{\mathbf{e}} > 0$ *on* Γ_{shock};

(e) *For any* $T \in (T_{P_1}, T_{P_2})$,

$$\phi_{\boldsymbol{\tau\tau}}(f_{\mathbf{e}}(T), T) < 0 \quad \Longleftrightarrow \quad f''_{\mathbf{e}}(T) > 0,$$

and

$$\phi_{\boldsymbol{\tau\tau}}(f_{\mathbf{e}}(T), T) > 0 \quad \Longleftrightarrow \quad f''_{\mathbf{e}}(T) < 0.$$

In [21], we provide a framework for the convexity of transonic shocks in the self-similar coordinates. Specifically, for the transonic shocks in the shock reflection-diffraction configurations, we have the following theorem.

Theorem 4.1 (Convexity of transonic shocks)**.** *If a solution of the shock reflection-diffraction problem is admissible in the sense of Definition 3.1, then its shock curve* Γ_{shock} *is strictly convex in the following sense: For any* $\mathbf{e} \in Con$, $f_{\mathbf{e}}$ *from Lemma 4.1 is concave on* (T_{P_1}, T_{P_2}), *and* $f''_{\mathbf{e}}(T) < 0$ *for all* $T \in (T_{P_1}, T_{P_2})$. *That is,* Γ_{shock} *is uniformly convex on closed subsets of its relative interior. Moreover, for a regular reflection solution in the sense of Definition 2.2 with pseudo-potential* $\varphi \in C^{0,1}(\Lambda)$ *satisfying Definition 3.1(i)–(iv), the shock is strictly convex if and only if Definition 3.1(v) holds.*

We remark that the strict convexity of the reflected-diffracted transonic shocks for the attached case when $u_1 > c_1$ and $\theta_{\mathrm{w}} = \theta_{\mathrm{w}}^{\mathrm{c}}$ can also be proved (see [21]).

Now we discuss the techniques developed in [21] by giving the main steps in the proof of Theorem 4.1. While the argument in [21] is for a general domain Ω, we focus here on the regular shock reflection-diffraction configurations, in which both the solution domain Ω and the solution structure are somewhat simpler.

Outline of the Proof of Theorem 4.1: The proof consists of eight steps, while the first three steps are general properties of shock reflection-diffraction solutions; see [19]. Below we use notation $\phi := \varphi - \varphi_1$.

1. We establish a relation between the extrema of the solution and the geometric shape of the transonic shock. For a fixed unit vector $\mathbf{e} \in \mathbb{R}^2$, denote $w := \partial_{\mathbf{e}}\phi$ in Ω. We show that, if a local minimum (or maximum) of w is attained at $P \in \Gamma^0_{\text{shock}}$ and $\boldsymbol{\nu}(P) \cdot \mathbf{e} < 0$, then $\phi_{\boldsymbol{\tau\tau}} > 0$ (or $\phi_{\boldsymbol{\tau\tau}} < 0$) at P, where $\boldsymbol{\nu}$ denotes the interior unit normal on Γ_{shock} towards Ω.

2. We establish a nonlocal relation between the values of $\phi_{\mathbf{e}}$ and the positions where these values are taken. Let ϕ be a solution as in Theorem 4.1, and let $\mathbf{e} \in Con$. We use the coordinates from Lemma 4.1. Assume that, for two different points $P = (T, f_{\mathbf{e}}(T))$ and $P_1 = (T_1, f_{\mathbf{e}}(T_1))$ on Γ_{shock},

$$f_{\mathbf{e}}(T) > f_{\mathbf{e}}(T_1) + f'_{\mathbf{e}}(T)(T - T_1), \qquad f'_{\mathbf{e}}(T) = f'_{\mathbf{e}}(T_1).$$

Then

(i) $d(P) := \mathrm{dist}(O_0, L_P) > \mathrm{dist}(O_0, L_{P_1}) =: d(P_1)$, where O_0 is the center of sonic circle of state (0), and L_P and L_{P_1} are the tangent lines of Γ_{shock} at P and P_1, respectively.

(ii) If the unit vector $\mathbf{e} \in Con$, then

$$\phi_{\mathbf{e}}(P) > \phi_{\mathbf{e}}(P_1).$$

3. We now develop a minimal/maximal chain argument. Let ϕ be an admissible solution, and let $\mathbf{e} \in \mathbb{R}^2$. Note that $\phi_{\mathbf{e}}$ satisfies the strong maximum principle in Ω. Then we can introduce the mininal (or maximal) chain as follows:

Let $E_1, E_2 \in \partial\Omega$. We say that points E_1 and E_2 are connected by a minimal (resp. maximal) chain with radius r if and only if there exist $r > 0$, integer $k_1 \geq 1$, and a chain of balls $\{B_r(C^i)\}_{i=0}^{k_1}$ such that

(i) $C^0 = E_1$, $C^{k_1} = E_2$, and $C^i \in \overline{\Omega}$ for $i = 0, \ldots, k_1$;

(ii) $C^{i+1} \in \overline{B_r(C^i) \cap \Omega}$ for $i = 0, \ldots, k_1 - 1$;

(iii) $\phi_{\mathbf{e}}(C^{i+1}) = \min_{\overline{B_r(C^i)\cap\Omega}} \phi_{\mathbf{e}} < \phi_{\mathbf{e}}(C^i)$ (resp. $\phi_{\mathbf{e}}(C^{i+1}) = \max_{\overline{B_r(C^i)\cap\Omega}} \phi_{\mathbf{e}} > \phi_{\mathbf{e}}(C^i)$) for $i = 0, \ldots, k_1 - 1$;

(iv) $\phi_{\mathbf{e}}(C^{k_1}) = \min_{\overline{B_r(C^{k_1})\cap\Omega}} \phi_{\mathbf{e}}$ (resp. $\phi_{\mathbf{e}}(C^{k_1}) = \max_{\overline{B_r(C^{k_1})\cap\Omega}} \phi_{\mathbf{e}}$).

For such a chain $\{C^i\}_{i=0}^{k_1}$, we also use the following terminology: The chain starts at E_1 and ends at E_2, or the chain is from E_1 to E_2.

This definition does not rule out the possibility that $B_r(C^i) \cap \partial\Omega \neq \emptyset$, or even $C^i \in \partial\Omega$, for some or all $i = 0, \ldots, k_1 - 1$. The radius r is a parameter in the definition of minimal or maximal chains. We do not fix r at this point. In fact, the radii are determined for various chains, respectively.

Then we prove the following results:

(a) *The chains with sufficiently small radius are connected sets.* More precisely, there exists r^* depending only on (ρ_0, ρ_1, γ) such that, for any minimal or maximal chain $\{C^i\}_{i=0}^{k_1}$ with $r \in (0, r^*]$, $\cup_{i=0}^{k_1} B_r(C^i) \cap \Omega$ is connected.

(b) *The existence of the minimal or maximal chain of radius $r < r^*$.* More precisely, if $E_1 \in \partial\Omega$, and E_1 is not a local minimum point (resp. maximum point) of $\phi_{\mathbf{e}}$ with respect to $\overline{\Omega}$, then, for any $r \in (0, r^*)$, there exists a minimal (resp. maximal) chain $\{G^i\}_{i=0}^{k_1}$ for $\phi_{\mathbf{e}}$ of radius r, starting at E_1, *i.e.* $G^0 = E_1$. Moreover, $G^{k_1} \in \partial\Omega$ is a local minimum (resp. maximum) point of $\phi_{\mathbf{e}}$ with respect to $\overline{\Omega}$, and $\phi_{\mathbf{e}}(G^{k_1}) < \phi_{\mathbf{e}}(E_1)$ (resp. $\phi_{\mathbf{e}}(G^{k_1}) > \phi_{\mathbf{e}}(E_1)$).

(c) *The minimal and maximal chains do not intersect.* Specifically, for any $\delta > 0$, there exists $r_1^* \in (0, r^*]$ such that the following holds: Let $\mathcal{C} \subset \partial\Omega$ be connected, let E_1 and E_2 be the endpoints of $\mathcal{C}$, and let there be a minimal chain $\{E^i\}_{i=0}^{k_1}$ of radius $r_1 \in (0, r_1^*]$, which starts at E_1 and ends at E_2. If there exists $H_1 \in \mathcal{C}^0 = \mathcal{C} \setminus \{E_1, E_2\}$ such that

$$\phi_{\mathbf{e}}(H_1) \geq \phi_{\mathbf{e}}(E_1) + \delta,$$

then, for any $r_2 \in (0, r_1]$, any maximal chain $\{H^j\}_{j=0}^{k_2}$ of radius r_2 starting from H_1 satisfies $H^{k_2} \in \mathcal{C}^0$, where $\mathcal{C}^0$ denotes the relative interior of curve $\mathcal{C}$ as before.

Note that, if H_1 is not a local maximum point of $\phi_{\mathbf{e}}$ with respect to $\overline{\Omega}$, then the existence of the maximal chain $\{H^j\}_{j=0}^{k_2}$ of radius r_2 starting from H_1 follows from result (b).

(d) *Result (c) also holds if the roles of minimal and maximal chains are interchanged.* For any $\delta > 0$, there exists $r_1^* \in (0, r^*]$ such that the following holds: Let $\mathcal{C} \subset \partial\Omega$ be connected, and let E_1 and E_2 be the endpoints of $\mathcal{C}$. Assume that there exists a maximal chain $\{E^i\}_{i=0}^{k_1}$ of radius $r_1 \in (0, r_1^*]$, which starts

at E_1 and ends at E_2. If there exists $H_1 \in \mathcal{C}^0$ such that
$$\phi_{\mathbf{e}}(H_1) \le \phi_{\mathbf{e}}(E_1) - \delta,$$
then, for any $r_2 \in (0, r_1]$, any minimal chain $\{H^j\}_{j=0}^{k_2}$ of radius r_2, starting from H_1, satisfies that $H^{k_2} \in \mathcal{C}^0$.

(e) *Two minimal chains do not intersect*: For any $r_1 \in (0, r^*]$, there exists $r_2^* = r_2^*(r_1) \in (0, r^*]$ such that the following holds: Let $\mathcal{C} \subset \partial\Omega$ be connected, and let E_1 and E_2 be the endpoints of $\mathcal{C}$. Assume that there exists a minimal chain $\{E^i\}_{i=0}^{k_1}$ of radius $r_1 \in (0, r^*]$, which starts at E_1 and ends at E_2. If there exists $H_1 \in \mathcal{C}^0$ such that
$$\phi_{\mathbf{e}}(H_1) < \phi_{\mathbf{e}}(E_2),$$
then, for any $r_2 \in (0, r_2^*]$, any minimal chain $\{H^j\}_{j=0}^{k_2}$ of radius r_2, starting from H_1, satisfies that $H^{k_2} \in \mathcal{C}^0$.

4. We use $\mathbf{e} \in Con$ that is sufficiently close to $\mathbf{e}_{\xi_2}$ for the following four steps below. We work in the corresponding (S, T)–coordinates so that it suffices to prove that the graph is concave:
$$f''_{\mathbf{e}}(T) \le 0 \qquad \text{for all } T \in (T_{P_1}, T_{P_2}).$$

If there exists $\hat{P} \in \Gamma^0_{\text{shock}}$ with $f''_{\mathbf{e}}(T_{\hat{P}}) > 0$, we prove the existence of a point $C \in \Gamma^0_{\text{shock}}$ such that $f''_{\mathbf{e}}(T_C) \ge 0$, and C is a point of strict local minimum of $\phi_{\mathbf{e}}$ along Γ_{shock} but is *not* a local minimum point of $\phi_{\mathbf{e}}$ relative to $\overline{\Omega}$.

5. Then we prove that there exists $C_1 \in \Gamma^0_{\text{shock}}$ such that there is a minimal chain with radius r_1 from C to C_1.

6. We show that the existence of points C and C_1 described above yields a contradiction (which implies that there is no $\hat{P} \in \Gamma^0_{\text{shock}}$ with $f''_{\mathbf{e}}(T_{\hat{P}}) > 0$). This is proved by showing the following facts:

- Let A_2 be a maximum point of $\phi_{\mathbf{e}}$ along Γ_{shock} lying between points C and C_1. Then A_2 is a local maximum point of $\phi_{\mathbf{e}}$ relative to Ω, and there is no point between C and C_1 such that the tangent line at this point is parallel to the one at A_2.
- Between C and A_2, or between C_1 and A_2, there exists a local minimum point C_2 of $\phi_{\mathbf{e}}$ along Γ_{shock} such that $C_2 \ne C_1$, or $C_2 \ne C$, and C_2 is not a local minimum point of $\phi_{\mathbf{e}}$ relative to domain $\overline{\Omega}$.
- Then, following a similar argument for Step 6, we arrive at a contradiction.

These indicate that $f''_{\mathbf{e}} \le 0$ on Γ_{shock}; that is, Γ_{shock} is convex. In the rest of the argument, we prove that $f''_{\mathbf{e}} < 0$ on Γ^0_{shock}.

7. We show that the shock graph is real analytic and then, for every $P \in \Gamma^0_{\text{shock}}$, either $f''_{\mathbf{e}}(T_P) < 0$ or there exists an even integer $k > 2$ such that $f^{(i)}_{\mathbf{e}}(T_P) = 0$ for all $i = 2, \dots, k-1$, and $f^{(k)}_{\mathbf{e}}(T_P) < 0$. This shows the strict convexity of the shock, which implies that the shock does not contain any straight segment. The above property is equivalent to the facts that $\partial^i_{\boldsymbol{\tau}}\phi(P) = 0$ for all $i = 2, \dots, k-1$, and $\partial^k_{\boldsymbol{\tau}}\phi(P) > 0$.

8. We show the uniform convexity of Γ^0_{shock} in the sense that $f''_{\mathbf{e}}(T_P) < 0$ for every $P \in \Gamma^0_{\text{shock}}$, or equivalently, $f''_{\mathbf{e}}(T) < 0$ on (T_{P_1}, T_{P_2}), for some (and thus any) $\mathbf{e} \in Con$. In fact, if it is not true, *i.e.* if $\phi_{\boldsymbol{\tau\tau}} = 0$ at some P_d, then we can obtain a contradiction by proving that there exists a unit vector $\mathbf{e} \in \mathbb{R}^2$ such that P_d is a

local minimum point of $\phi_{\mathbf{e}}$ along Γ^0_{shock}, but P_d is not a local minimum point of $\phi_{\mathbf{e}}$ in Ω. Then we can construct a minimal chain for $\phi_{\mathbf{e}}$ connecting P_{d} to $C^{k_1} \in \partial\Omega$. We show that

- $C^{k_1} \notin \Gamma_{\text{sonic}}$,
- $C^{k_1} \notin \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}$,
- $C^{k_1} \notin \Gamma_{\text{shock}}$.

This implies that $\phi_{\boldsymbol{\tau\tau}} > 0$ on Γ^0_{shock} so that $f''_{\mathbf{e}}(T) < 0$ on (T_{P_1}, T_{P_2}); see Lemma 4.1.

5. **Uniqueness and Stability of Regular Shock Reflection-Diffraction Configurations.** In this section, we discuss the uniqueness and stability of global regular shock reflection-diffraction configurations. More specifically, we describe the results in Chen-Feldman-Xiang [22].

As indicated earlier, recent results [24, 25, 33, 46] have shown the non-uniqueness of solutions with planar shocks in the class of entropy solutions with shocks of the Cauchy problem for the multidimensional compressible Euler system. Moreover, the uniqueness problem for general self-similar solutions of the Euler system is still open (*cf.* [24]). While these results do not apply directly to our case, they indicate that it be natural to study the uniqueness of solutions in some more restrictive class, instead of general time-dependent solutions (*i.e.* solutions of Problem 2.1), or even general self-similar solutions as in Definition 2.2.

In [22], we have established the uniqueness of regular reflection solutions for each wedge angle in the class of *admissible solutions* introduced in Definition 3.1.

Theorem 5.1 (Uniqueness)**.** *For any wedge angle $\theta_{\text{w}} \in (\theta^{\text{d}}_{\text{w}}, \frac{\pi}{2})$ when $u_1 \le c_1$ and $\theta_{\text{w}} \in (\theta^{\text{c}}_{\text{w}}, \frac{\pi}{2})$ when $u_1 > c_1$, any solution, satisfying both properties* (i)–(iv) *in Definition* 3.1 *and one of the following properties*:

(a) *the transonic shock Γ_{shock} is convex, i.e. domain Ω is a convex set,*

(b) *condition* (3.5) *holds,*

is unique in the class of admissible solutions. Moreover, such solutions are continuous with respect to the wedge angle θ_{w} in the C^1–norm (*more precisely, the continuity with respect to the norm described in Remark* 5.1 *below*)*.*

Remark 5.1. For an admissible solution φ with a wedge angle θ_{w}, we define its norm based on its restriction to Ω. Since region Ω depends on the solution, we map a unit square $Q^{\text{iter}} = (0,1)^2$ to Ω and use this mapping to define a function u on Q^{iter}, which corresponds to $\varphi_{|\Omega}$. Furthermore, the sides of square Q^{iter} are mapped to the boundary parts Γ_{sonic}, Γ_{wedge}, Γ_{sym}, and Γ_{shock}. The mapping depends on $(\varphi, \theta_{\text{w}})$ and is invertible; that is, given a function u on Q^{iter} and θ_{w}, we can recover φ and Ω. Moreover, this mapping and its inverse have appropriate continuity properties. See [19, §12.2 and §17.2] for the details. Then we define function spaces for admissible solutions and "approximate admissible solutions" in terms of the function spaces for the corresponding functions u on Q^{iter}. The convergence of admissible solutions $\varphi^{(i)} \to \varphi^{(\infty)}$ in the C^1–norm as the corresponding wedge angles $\theta^{(i)}_{\text{w}} \to \theta^{(\infty)}_{\text{w}}$, defined in terms of convergence in an appropriate norm for the functions on Q^{iter}, implies

$$
\begin{aligned}
&\|\varphi^{(i)}\|_{C^1(\Omega^{(i)})} \le C \qquad \text{for all } i,\\
&\|\varphi^{(i)} - \varphi^{(\infty)}\|_{C^1(\overline{\Omega^{(i)} \cap \Omega^{(\infty)}})} + d_H(\overline{\Omega^{(i)}}, \overline{\Omega^{(\infty)}}) \to 0 \qquad \text{as } \theta^{(i)}_{\text{w}} \to \theta^{(\infty)}_{\text{w}},
\end{aligned}
\tag{5.1}
$$

where d_H denotes the Hausdorff distance between the sets.

Remark 5.2. By Theorem 4.1, conditions (a) and (b) in Theorem 5.1 for the solutions satisfying properties (i)–(iv) in Definition 3.1 are equivalent.

Remark 5.3. We note that, under either of conditions (a) and (b) in Theorem 5.1, the solution is an admissible solution. Indeed, in both cases, the solution satisfies properties (i)–(iv) in Definition 3.1. If, in addition, condition (b) holds, then the solution is admissible directly from Definition 3.1. Remark 5.2 shows the same for the case when condition (a) holds.

The proof of Theorem 5.1 is obtained by showing the following proposition on the existence and uniqueness of a family of admissible solutions that are continuous with respect to $\theta_{\rm w}$, containing a given admissible solution.

Proposition 5.1. *Fix (ρ_0, ρ_1, γ). Define interval $I := (\theta_{\rm w}^{\rm d}, \frac{\pi}{2}]$ when $u_1 \le c_1$ and $I := (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$ when $u_1 > c_1$. For every admissible solution φ^* with a wedge angle $\theta_{\rm w}^* \in I$, there exists a family*

$$\mathfrak{S} = \{(\varphi, \theta_{\rm w}) \ : \ \theta_{\rm w} \in I, \ \varphi \in C^{0,1}(\Lambda(\theta_{\rm w}))\}$$

such that

$$(\varphi^*, \theta_{\rm w}^*) \in \mathfrak{S}, \tag{5.2}$$

and $\mathfrak{S}$ satisfies the following properties:

(a) *For each $\theta_{\rm w} \in I$, there exists one and only one pair $(\varphi, \theta_{\rm w}) \in \mathfrak{S}$. Then we can define $\varphi^{(\theta_{\rm w})} := \varphi$ if $(\varphi, \theta_{\rm w}) \in \mathfrak{S}$.*

(b) *Each $\varphi^{(\theta_{\rm w})}$ is an admissible solution corresponding to the wedge angle $\theta_{\rm w}$.*

(c) *$\varphi^{(\frac{\pi}{2})}$ is the normal shock reflection solution* (see §3.1 in [17] for the definition).

(d) *$\varphi^{(\theta_{\rm w})}$ is continuous with respect to the wedge angle $\theta_{\rm w} \in I$ in the C^1–norm as in Remark 5.1.*

Moreover, a family $\mathfrak{S}$ satisfying properties (a)–(d) listed above (but without requiring (5.2)) is unique. That is, if there are two families $\mathfrak{S}_1$ and $\mathfrak{S}_2$ satisfying properties (a)–(d), then $\mathfrak{S}_1 = \mathfrak{S}_2$. Thus, the family $\mathfrak{S}$ contains all the admissible solutions for all $\theta_{\rm w} \in I$.

Proposition 5.1 directly implies Theorem 5.1.

Proposition 5.1 is proved by showing the local uniqueness and existence of admissible solutions.

As we have discussed in the introduction, the outline of the uniqueness proof (*i.e.* Proposition 5.1) is the following: If there are two different admissible solutions, defined by the potential functions φ and $\hat{\varphi}$, for some wedge angle $\theta_{\rm w}^* \in I \setminus \{\frac{\pi}{2}\}$, it suffices to:

(i) construct continuous families of solutions parametrized by the wedge angle $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$, starting from φ and $\hat{\varphi}$, respectively, in the norm discussed in Remark 5.1;

(ii) prove *local uniqueness*: If two admissible solutions with the same wedge angle are close in the norm given in the second line of (5.1), then they are equal.

Combining this with the fact that, by Remark 3.1, both families converge to the normal reflection as $\theta_{\rm w} \to \frac{\pi}{2}-$, we obtain a contradiction; see more details in §5.3 below. Furthermore, the continuous family defined above can be extended to all $\theta_{\rm w} \in I$, hence determining the family $\mathfrak{S}$ in Proposition 5.1.

In order to construct the continuous family of solutions $\mathfrak{S}$ described in Proposition 5.1, starting from the given solution $\varphi = \varphi^{(\theta_{\rm w}^*)}$, it suffices to show that any given admissible solution can be perturbed, that is, an admissible solution can be constructed to be close to φ for all wedge angles which are sufficiently close to $\theta_{\rm w}^*$. More precisely, using the mapping of admissible solutions to the functions on the unit square discussed in Remark 5.1, we work in an appropriately weighted and scaled $C^{2,\alpha}$ space on $Q^{\rm iter}$. We choose this function space according to the norms and the other quantities in the *a priori* estimates for admissible solutions in [19], mapped to $Q^{\rm iter}$. Denote the norm in this space by $\|\cdot\|^*$. Thus, we consider space $C_*(Q^{\rm iter})$, which is completion of $C^\infty(\overline{Q^{\rm iter}})$ with respect to norm $\|\cdot\|^*$. This space satisfies

$$C_*(Q^{\rm iter}) \subset C^{1,\alpha}(\overline{Q^{\rm iter}}) \cap C^{2,\alpha}(Q^{\rm iter}).$$

For any admissible solution φ, the corresponding function u on $Q^{\rm iter}$ satisfies $u \in C_*(Q^{\rm iter})$. Now we state the local existence assertion.

Proposition 5.2 (Local existence)**.** *Fix any admissible solution $(\hat{\varphi}, \hat{\theta}_{\rm w})$ with $\hat{\theta}_{\rm w} \in I$. Then, for every sufficiently small $\varepsilon > 0$, there is $\delta > 0$ with the following property: For each $\theta_{\rm w} \in [\hat{\theta}_{\rm w} - \delta, \hat{\theta}_{\rm w} + \delta)] \cap I$, there exists an admissible solution φ such that u and $\hat{u}$ on $Q^{\rm iter}$ corresponding to φ and $\hat{\varphi}$, respectively, satisfy*

$$\|u - \hat{u}\|^* < \varepsilon.$$

Note that, if ε is sufficiently small, the solutions obtained in Proposition 5.2 are unique for each wedge angle, by the local uniqueness.

Thus, to prove Proposition 5.1, it suffices to prove the local uniqueness, as well as the local existence in the sense of Proposition 5.2. See §5.3 below for more details in the proof of Proposition 5.2 from these properties. In fact, from Remark 3.3, we study these questions for the free boundary problem (2.5) and (3.6)–(3.7), where the unknowns are φ in Ω and $\Gamma_{\rm shock}$. Moreover, the admissible solutions satisfy property (ii) in Definition 3.1, from which equation (2.5) is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{\rm sonic}}$ in the supersonic and sonic cases $|D\varphi_2(P_0)| \geq c_2$ and uniformly elliptic in $\overline{\Omega}$ in the subsonic case $|D\varphi_2(P_0)| < c_2$.

The proofs of the local existence and uniqueness are different for the following two cases:

(a) Supersonic and subsonic-near-sonic case: $|D\varphi_2(P_0)| > (1-\sigma)c_2$,
(b) Subsonic-away-from-sonic case: $|D\varphi_2(P_0)| \leq (1-\sigma)c_2$,

where $\sigma > 0$ depends on (ρ_0, ρ_1, γ) and is such that, for the wedge angles satisfying $(1-\sigma)c_2 \leq |D\varphi_2(P_0)| \leq 1$ (which are the subsonic-near-sonic and sonic cases), the admissible solutions are $C^{2,\alpha}$ up to P_0 according to [19].

The reason for the different proofs for cases (a) and (b) is that, in the supersonic and sonic case, the degenerate ellipticity of equation (2.5) near $\Gamma_{\rm sonic}$ or P_0 makes it difficult to use the linearization of problem (2.5) and (3.6)–(3.7) for the application of the implicit function theorem which would imply both the local existence and uniqueness. On the other hand, under the conditions stated in case (a), φ is $C^{1,1}$ up to $\Gamma_{\rm sonic}$ in the supersonic case (by Theorem 3.2(i)), and C^2 up to P_0 in the subsonic-near-sonic and sonic cases; this higher regularity allows us to use the different methods described below. In the subsonic-away-from-sonic case (b), the known regularity up to P_0 is $C^{1,\alpha}$, *i.e.* lower than that in case (a), but equation (2.5) is uniformly elliptic in $\overline{\Omega}$; this allows to analyze the linearization of problem (2.5) and

(3.6)–(3.7) at φ, and thus obtain the local uniqueness and existence by the implicit function theorem.

It remains to discuss the proof of the local uniqueness and existence in the supersonic and subsonic-near-sonic case (a). The outline of this proof is in §5.1–§5.2 below.

5.1. **Local uniqueness in the supersonic and subsonic-near-sonic case (a).** Assume that φ and φ^* are regular shock reflection solutions for the same wedge angle $\theta_{\rm w}$, which are $C^{1,1}$ up to $\overline{\Gamma_{\rm sonic}}$ (where we denote $\overline{\Gamma_{\rm sonic}} = \{P_0\}$ in the subsonic and sonic cases) and satisfying the properties listed in Theorem 5.1. Let Ω and Ω^* be respectively their elliptic regions, and let $\Gamma_{\rm shock}$ and $\Gamma^*_{\rm shock}$ be respectively their reflected shocks. We recall that φ and φ^* satisfy (2.5) and (3.6)–(3.7) in Ω and Ω^*, respectively.

Let $\hat{\Omega} := \Omega \cap \Omega^*$, and let $\hat{\Gamma}_{\rm shock} := \partial\hat{\Omega} \cap (\Gamma^*_{\rm shock} \cup \Gamma_{\rm shock})$. We now show that, under the following assumption:

$$\|\varphi - \varphi^*\|_{C^1(\hat{\Omega})} + \|\varphi - \varphi_1\|_{C^0((\Omega\cup\Omega^*)\setminus\hat{\Omega})} + \|\varphi^* - \varphi_1\|_{C^0((\Omega\cup\Omega^*)\setminus\hat{\Omega})} \le \delta_2, \tag{5.3}$$

the function, $\delta\varphi := \varphi - \varphi^*$, satisfies the boundary condition:

$$\mathcal{M}(\delta\varphi) = \beta_{\boldsymbol\nu}(\delta\varphi)_{\boldsymbol\nu} + \beta_{\boldsymbol\tau}(\delta\varphi)_{\boldsymbol\tau} + \vartheta\delta\varphi = 0 \qquad \text{on the inner shock } \hat{\Gamma}_{\rm shock}, \tag{5.4}$$

with

$$\beta_{\boldsymbol\nu} > 0, \qquad \vartheta < 0, \tag{5.5}$$

where $\boldsymbol\nu$ is the unit inner normal and $\boldsymbol\tau$ is the unit tangent on $\hat{\Gamma}_{\rm shock}$. We note that the property that $\vartheta < 0$ in (5.5) is obtained using the convexity of $\Gamma^*_{\rm shock}$ and $\Gamma_{\rm shock}$.

Also, it follows from [18] that $\delta\varphi$ satisfies a homogeneous linear elliptic equation in $\hat{\Omega}$ for which the comparison principles hold. Properties (5.5), combined with methods of [18], show that Hopf's lemma holds for $\delta\varphi$ on $\hat{\Gamma}_{\rm shock}$. Finally, $\delta\varphi$ satisfies the homogeneous Neumann condition on $(\partial\hat{\Omega} \cap \partial\Lambda) \setminus \{P_3\}$, and $\delta\varphi = 0$ on $\Gamma_{\rm sonic}$.

These facts ensure that $\delta\varphi \equiv 0$ in $\hat{\Omega}$. From this, we can show

$$\Omega^* = \Omega, \qquad \varphi = \varphi^* \quad \text{in } \Omega. \tag{5.6}$$

This completes the proof of the local uniqueness.

Remark 5.4. We remark that, due to the issue that the regularity of φ at the reflection point P_0 is only $C^{1,\alpha}$ for the subsonic-away-sonic reflection case $|D\varphi_2(P_0)| \le (1-\sigma)c_2$, we cannot apply this argument. However, as we discussed earlier, the implicit function theorem can be applied in that case.

5.2. **Local existence in the supersonic and subsonic-near-sonic case (a).** Now we discuss the proof of the local existence, Proposition 5.2. The existence of a solution is obtained by the application of the Leray-Schauder degree theory [55, §13.6($A4^*$)]; see also [19, §3.4].

In order to apply the degree theory, the iteration set should be bounded and open in an appropriate function space (in fact, in its product with the parameter space, *i.e.* interval $[\hat{\theta}_{\rm w} - \delta, \hat{\theta}_{\rm w} + \delta)] \cap I$ of the wedge angles), the iteration map should be defined and continuous on the closure of the iteration set, and any fixed point of the iteration map should not occur on the boundary of the iteration set. We choose this function space according to the norms and the other quantities in the *a priori* estimates. Moreover, since we have to use the same function space for all values of the parameters, and the functions require to have the same domain, we define the iteration set in terms of the functions on the unit square $Q^{\rm iter}$, which are related to the admissible solutions by the mapping described in Remark 5.1. The function space is $C_*(Q^{\rm iter})$, introduced above. Let $\hat{u}$ be the function on $Q^{\rm iter}$ corresponding to the admissible solution $\hat{\varphi}$ for the wedge angle $\hat{\theta}_{\rm w}$ in Proposition 5.2. In order to prove the existence result in Proposition 5.2 for given ε and δ, we define the iteration set by

$$\mathcal{K}_{\varepsilon,\delta}^{(\hat{u},\hat{\theta}_{\rm w})} := \{(u,\theta_{\rm w}) \in C_*(Q^{\rm iter}) \times ([\hat{\theta}_{\rm w}-\delta,\ \hat{\theta}_{\rm w}+\delta]\cap I) \ \ : \ \ \|u-\hat{u}\|^* < \varepsilon\}. \quad (5.7)$$

From its definition, the iteration set is non-empty, open (in the subspace topology) and bounded in $C_*(Q^{iter}) \times ([\hat{\theta}_w - \delta,\ \hat{\theta}_w + \delta]\cap I)$.

We also define the iteration set for each wedge angle $\theta_{\rm w} \in [\hat{\theta}_{\rm w}-\delta,\ \hat{\theta}_{\rm w}+\delta]\cap I$ by

$$\mathcal{K}_{\varepsilon}^{(\hat{u},\hat{\theta}_{\rm w})}(\theta_{\rm w}) := \{u \in C_*(Q^{\rm iter}) \ \ : \ \ (u,\theta_{\rm w}) \in \mathcal{K}_{\varepsilon,\delta}^{(\hat{u},\hat{\theta}_{\rm w})}\}. \quad (5.8)$$

To prove Proposition 5.2, we need to show the existence of an admissible solution in $\mathcal{K}_{\varepsilon,\theta_{\rm w}}^{(\hat{u},\hat{\theta}_{\rm w})}$ for each $\theta_{\rm w} \in [\hat{\theta}_{\rm w}-\delta,\ \hat{\theta}_{\rm w}+\delta]\cap I$ if ε is small depending on $(\rho_0,\rho_1,\gamma,\hat{\theta}_{\rm w})$, and δ is small depending on ε and $(\rho_0,\rho_1,\gamma,\hat{\theta}_{\rm w})$.

The iteration map $\mathcal{F}$ is defined as follows:

Given $(u,\theta_{\rm w}) \in \overline{\mathcal{K}_{\varepsilon,\delta}^{(\hat{u},\hat{\theta}_{\rm w})}}$, define the corresponding *elliptic domain* $\Omega = \Omega(u,\theta_{\rm w})$ by mapping from the unit square $Q^{\rm iter}$ to the *physical plane*, as discussed in Remark 5.1. This determines iteration $\Gamma_{\rm shock}$ and function φ in Ω, depending on $(u,\theta_{\rm w})$. We set up a boundary value problem in Ω for a *new iteration potential* $\tilde{\varphi}$ by modifying problem (2.5) and (3.6)–(3.7), by partially substituting φ into the coefficients of (2.5), and making other modifications including the ellipticity cutoff in the equation.

In the supersonic and sonic cases, the modified equation is elliptic in $\overline{\Omega}\setminus\Gamma_{\rm sonic}$, degenerate near $\Gamma_{\rm sonic}$ (or P_0 in the sonic case), and nonlinear near $\Gamma_{\rm sonic}$. In the subsonic case, the modified equation is linear and uniformly elliptic in $\overline{\Omega}$.

In all the supersonic, sonic, and subsonic cases, we prescribe one condition on $\Gamma_{\rm shock}$, which is an oblique derivative condition, by combining the two conditions in (3.6) and partially substituting φ into the coefficients of the main terms.

Let $\tilde{\varphi}$ be the solution of the boundary value problem in Ω. We show that $\tilde{\varphi}$ gains the regularity in comparison with φ. Then we define $\tilde{u}$ on $Q^{\rm iter}$ by mapping $\tilde{\varphi}$ back in such a way that the gain-in-regularity of the solution is preserved, which is needed in order to have the compactness of the iteration map. This requires some care, since the original mapping between $Q^{\rm iter}$ and the *physical domain* is defined by u and hence has a lower regularity. Then the iteration map is defined by

$$\mathcal{F}(u,\theta_{\rm w}) = \tilde{u}.$$

The boundary value problem in the definition of $\mathcal{F}$ is defined so that, at the fixed point $u=\tilde{u}$, its solution satisfies the potential flow equation (2.5) with the ellipticity cutoff in a small neighborhood of $\Gamma_{\rm sonic}$ in the supersonic case, both the Rankine-Hugoniot conditions (3.6) on $\Gamma_{\rm shock}$, and the boundary condition (3.7) on $\Gamma_{\rm wedge}\cup\Gamma_{\rm sym}$. On the sonic arc $\Gamma_{\rm sonic}$ in the supersonic case and at P_0 in the subsonic and sonic cases, we need two conditions: $\tilde{\varphi}=\varphi_2$ and $D\tilde{\varphi}=D\varphi_2$. However, we can prescribe only one condition on the fixed boundary. We choose the Dirichlet condition $\tilde{\varphi}=\varphi_2$ on $\Gamma_{\rm sonic}$ in the supersonic case and at P_0 in the subsonic and sonic cases, and prove that $D\tilde{\varphi}=D\varphi_2$ on $\Gamma_{\rm sonic}$ or at P_0 holds for the solution of the iteration problem for the fixed point.

Then we prove the following facts:

(i) Any fixed point $u = \mathcal{F}(u, \theta_{\rm w})$, mapped to the *physical plane*, is an admissible solution φ. For that, we remove the ellipticity cutoff and prove the inequalities and monotonicity properties in the definition of the admissible solutions for the regions and the wedge angles where they are not readily known from the definition of the iteration set.

(ii) The iteration map is continuous on $\overline{\mathcal{K}^{(\hat{u},\hat{\theta}_{\rm w})}_{\varepsilon,\delta}}$ and compact. We prove this by using the gain-in-regularity of the solution of the iteration boundary value problem.

(iii) Any fixed point of the iteration map cannot occur on the boundary of the iteration set if δ is small depending on ε and (ρ_0, ρ_1, γ). Now we discuss this step in more details:

The *small* iteration set (5.8) is the first key difference between this proof of the local existence and the proof of the existence of admissible solutions in [19], which is also obtained by the Leray-Schauder degree argument. In [19], the continuity of admissible solutions with respect to $\theta_{\rm w}$ was not studied; for this reason, the iteration set is chosen to be *large* for the wedge angles away from $\frac{\pi}{2}$. That is, the iteration set for such a wedge angle is defined by the bounds in the appropriate norms related to the *a priori* estimates and by the lower bounds of certain directional derivatives, corresponding to the strict monotonicity properties so that the actual solution cannot be on the boundary of the iteration set according to the *a priori* estimates. In the present case of *small* iteration set (5.8), a different approach is developed, based on the local uniqueness and compactness of admissible solutions shown in [19]. That is, fixing small $\varepsilon > 0$, and assuming that, for any $\delta > 0$, there exists an admissible solution $\tilde{\varphi}$ for the wedge angle $\tilde{\theta}_{\rm w}$ such that $|\tilde{\theta}_{\rm w} - \hat{\theta}_{\rm w}| \le \delta$ and $\|\tilde{u} - \hat{u}\|^* = \varepsilon$, we obtain a sequence of admissible solutions and their wedge angles $(\varphi^{(i)}, \theta^{(i)}_{\rm w})$ with $\theta^{(i)}_{\rm w} \to \hat{\theta}_{\rm w}$ and $\|u^{(i)} - \hat{u}\|^* = \varepsilon$. Then, using the compactness of admissible solutions, we can send to a limit for a subsequence so that an admissible solution $\bar{\varphi}$ is obtained for the wedge angle $\hat{\theta}_{\rm w}$ such that $\|\bar{u} - \hat{u}\|^* = \varepsilon$. This contradicts the local uniqueness if ε is small.

Now the Leray-Schauder degree theory guarantees that the fixed point index:

$$\mathrm{Ind}(\mathcal{F}(\cdot, \theta_{\rm w}),\ \overline{\mathcal{K}^{(\hat{u},\hat{\theta}_{\rm w})}_{\varepsilon}(\theta_{\rm w})}) \tag{5.9}$$

of the iteration map on the iteration set (for given $\theta_{\rm w}$) is independent of the wedge angle $\theta_{\rm w} \in [\hat{\theta}_{\rm w} - \delta,\ \hat{\theta}_{\rm w} + \delta] \cap I$.

It remains to show that, at some wedge angle, index (5.9) is non-zero. We show that, for the wedge angle $\hat{\theta}_{\rm w}$,

$$\mathrm{Ind}(\mathcal{F}(\cdot, \hat{\theta}_{\rm w}),\ \overline{\mathcal{K}^{(\hat{u},\hat{\theta}_{\rm w})}_{\varepsilon}(\hat{\theta}_{\rm w})}) = 1.$$

We prove this by showing that

$$\mathcal{F}(v, \hat{\theta}_{\rm w}) = \hat{u} \qquad \text{for each } v \in \mathcal{K}^{(\hat{u},\hat{\theta}_{\rm w})}_{\varepsilon}(\hat{\theta}_{\rm w}). \tag{5.10}$$

This means that the iteration boundary value problem in domain $\Omega(v, \hat{\theta}_{\rm w})$ defined by every $v \in \mathcal{K}^{(\hat{u},\hat{\theta}_{\rm w})}_{\varepsilon}(\hat{\theta}_{\rm w})$ has the unique solution $\hat{\varphi}$ (in fact, its carefully defined extension from $\Omega(\hat{u}, \hat{\theta}_{\rm w})$). This step is another key difference from the existence proof of admissible solutions in [19]. In [19], the iteration set includes the normal reflection $\varphi^{\rm normal}$ for $\theta_{\rm w} = \frac{\pi}{2}$, and property (5.10) is shown for $\theta_{\rm w} = \frac{\pi}{2}$ and $u^{\rm normal}$ in the right-hand side. Since $\varphi^{\rm normal}$ is an explicitly known uniform state, globally defined, showing (5.10) is straightforward for the normal reflection, and does not require defining its extension, or any special properties of the coefficients of the iteration problem. In the present case, when $\hat{\varphi}$ is an arbitrary admissible solution, this step is much more involved, and requires an extension of $\hat{\varphi}$ from Ω to a larger region (so that the extension satisfies certain properties) and some careful definition of the coefficients of the iteration equation and the boundary condition on $\Gamma_{\rm shock}$, for which we need at least the $C^{1,1}$–regularity of φ near $\Gamma_{\rm sonic}$. Thus, our method works for the supersonic and subsonic-near-sonic case; however, it does not readily work for the subsonic-away-from-sonic case (for this reason, in this case, we use a different approach as we discussed above).

This completes the proof of the local existence of supersonic and subsonic-near-sonic reflection solutions.

5.3. **Proof of Proposition 5.1.** Based on the local uniqueness and existence, we employ the compactness of admissible solutions proved in [19] to conclude that, for every admissible solution φ^* with the wedge angle $\theta_{\rm w}^* \in I$, a family $\mathfrak{S}$ with properties listed in Proposition 5.1 exists.

It remains to prove the uniqueness of admissible solutions for each wedge angle.

For a given wedge angle $\theta_{\rm w}$ as in Theorem 5.1, assume that there are two admissible solutions φ and $\tilde{\varphi}$ corresponding to the wedge angle $\theta_{\rm w}^*$. Let $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ be the continuous families with $(\varphi, \theta_w^*) \in \mathfrak{S}$ and $(\tilde{\varphi}, \theta_w^*) \in \tilde{\mathfrak{S}}$ in Proposition 5.1. Let $\mathfrak{A}$ be the set of all $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$ such that $\varphi^{\theta_{\rm w}} = \tilde{\varphi}^{\theta_{\rm w}}$. Since $\frac{\pi}{2} \in \mathfrak{A}$ by (c) of Proposition 5.1, it follows that $\mathfrak{A} \neq \emptyset$. The continuity of both families $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ with respect to θ_w implies that $\mathfrak{A}$ is closed. Also, by the assumption above, $\theta_{\rm w}^* \notin \mathfrak{A}$. Denote $\theta_{\rm w}^{\rm inf} := \inf \mathfrak{A}$, then $\theta_{\rm w}^{\rm inf} \in (\theta_{\rm w}^*, \frac{\pi}{2}]$. Now, using the continuity of families $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$, we can show that, choosing $\theta_{\rm w} \in (\theta_{\rm w}^*, \theta_{\rm w}^{\rm inf})$ to be sufficiently close to $\theta_{\rm w}^{\rm inf}$, we obtain that $\varphi^{(\theta_{\rm w})} = \tilde{\varphi}^{(\theta_{\rm w})}$ by the local uniqueness property. This contradicts the definition of $\theta_{\rm w}^{\rm inf}$.

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